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Diffusive inflationary cosmology

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Abstract. *In the paradigm of inflation, at some time the early universe behaves roughly as the de Sitter solution of the relativistically covariant (i.e. hyperbolic) vacuum Einstein field equations with a large positive cosmological constant. Here, we consider what happens if a metric $g(t)$ on a compact 3-manifold evolves according to R. Hamilton's parabolic Ricci flow equation* $g'(t) = -2k \operatorname{Ric}(g(t)) + Dg(t)$*, with an added positive source term $Dg(t)$ which is analogous to the cosmological term in Einstein's equation. In spite of the fact that this Ricci-flow equation is parabolic, and hence nonrelativistic, we show that if the diffusion constant k in the Ricci-flow equation is related to the cosmological constant Λ by $\Lambda = 3/16\,(c/k)^2$ and $D = 1/2\, c^2/k$, then the Ricci flow solution agrees with the de Sitter solution, modulo exponentially decaying terms, in the case where spherical symmetry is assumed. In the case where the initial metric on the spatial 3-manifold has positive (but not necessarily nearly constant) Ricci curvature, we show that under the Ricci flow, the universe not only expands, but becomes as round and de Sitter-like as desired, by choosing k appropriately. In contrast, similar "no hair" theorems in the relativistic case, not only assume a homogeneous spatial metric,but also have weaker conclusions. Moreover, in the relativistic setting, most attempts to reduce local inhomogeneities, in order to conform with observations, are rather contrived, as their authors admit.*

1. INTRODUCTION

In the past decade, a number of differential geometers led by R. Hamilton have been active in the study of geometrical heat equations. In particular, there have been studies (e.g., see [7, 8, 9]) of the Ricci flow equation for a time-dependent metric $g(t)$ on a compact n-manifold M:

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(1) $$g'(t) = -2k\, Ric(g(t)) + D\, g(t),$$

where $Ric(g(t))$ is the Ricci tensor ($Ric_{ij} = R_{ij} = R^k{}_{ikj}$) of the metric $g(t)$, and k is a positive constant with the dimensions of the heat diffusivity constant, namely length^2/time and D is a constant with dimension 1/time. Hamilton (see [7] or [8]) routinely sets $k = 1$, and most often takes $D = 0$, or lets D be the time-dependent function $2k/n\, \overline{R}(t)$, where

$$\overline{R}(t) \equiv \left[\int_M \nu(g(t)) \right]^{-1} \int_M R(g(t))\, \nu(g(t))$$

is the average scalar curvature, so that the Ricci flow is normalized, in the sense that the volume of $(M, g(t))$ is constant. However, for the time being, let k and D arbitrary, and for simplicity, time-independent. Equation (1) is a nonlinear, weakly-parabolic system, in the sense that the characteristic matrix (or symbol) formed from the second-order coordinate derivatives of g_{ij} in $-2k\, Ric(g)$ has nonnegative (but not all positive) eigenvalues at each nonzero convector. Henceforth, we assume that M has dimension 3. In the 3 + 1-formulation of general relativity, the Einstein evolution equations form a weakly hyperbolic system. Indeed, the metric ${}^{(4)}g = -c^2 dt^2 + g(t)$ satisfies the vacuum Einstein field equations ${}^{(4)}R_{\mu\nu} - 1/2\, {}^{(4)}R\, {}^{(4)}g_{\mu\nu} + \Lambda\, {}^{(4)}g_{\mu\nu} = 0$ with cosmological constant Λ, if and only if the following weakly hyperbolic system of evolution equations and constraints are satisfied (e.g., see [3, 5, 14]) (where $(g' \times g')_{ik} \equiv g^{hj} g'_{ih} g'_{jk}$ and $Tr(g') \equiv g^{ij} g'_{ij}$)

(2) $$c^{-2} g''(t) = c^{-2} [g'(t) \times g'(t) - \frac{1}{2}\, Tr(g'(t))\, g'(t)]$$
$$- 2\, Ric(g(t)) + 2\, \Lambda g(t)$$

(3) $$\frac{1}{4} c^{-2}\, [Tr(g'(t))^2 - \| g'(t) \|^2] + R(g(t)) = 2\Lambda$$

(4) $$Div[g'(t) - Tr(g'(t))\, g(t)] = 0.$$

If the constraints (3) and (4) hold initially then they are preserved under the evolution (2) due to the invariance of the equation ${}^{(4)}R_{\mu\nu} - 1/2\, {}^{(4)}R\, {}^{(4)}g_{\mu\nu} + \Lambda\, {}^{(4)}g_{\mu\nu} = 0$ under space-time diffeomorphisms. The weakness of the parabolicity of (1) and the hyperbolicity of (2) is due to the invariance of these systems under the diffeomorphism group of the 3-dimensional Cauchy hypersurface M. In the case of (1), this weakness can be easily eliminated the following «DeTurck trick» (a term dubbed by Hamilton who attributes the idea to D.

DeTurck [4]). Let $\{\Gamma^k_{ij}(t)\}$ be the Levi-Civita connection of $g(t)$ and define a global vector field $V(t)$ on M via $V(t)^k = k\, g\,(t)^{ij}\,[\Gamma^k_{ij}(t) - \Gamma^k_{ij}(0)]$. Let φ_t be the flow on M induced by $V(t)$ (i.e. $d/dt\,\varphi_t(p) = V(t)_p$ for all $p \in M$). Let $\tilde{g}(t) = \varphi_t^* g(t)$. Then

$$
\begin{aligned}
(5)\qquad \tilde{g}'(t) &= \frac{d}{dt}[\varphi_t^* g(t)] = \mathcal{L}_{V(t)}\,\varphi_t^* g(t) + \varphi_t^* g'(t)\\
&= \mathcal{L}_{V(t)}\varphi_t^* g(t) - \varphi_t^*(2k\,Ric(g(t)) + Dg(t))\\
&= \mathcal{L}_{V(t)}\varphi_t^* g(t) - 2k\,Ric(\varphi_t^* g(t)) + D\varphi_t^* g(t)\\
&= \mathcal{L}_{V(t)}\tilde{g}(t) - 2k\,Ric(\tilde{g}(t)) + D\tilde{g}(t)
\end{aligned}
$$

This equation for $\tilde{g}(t)$ is strictly parabolic, as one can check that the Lie derivative term cancels some of the second derivative terms in $-2k\,Ric(\tilde{g}(t))$, so that the resulting top-order term of the right side is $k\tilde{g}^{ij}\,\partial^2\tilde{g}/\partial x^i \partial x^j$ in local coordinates. One can then use the standard theory of quasi-linear parabolic systems to solve (5) for $\tilde{g}(t)$ for short time $0 \leqslant t < \epsilon$, and then $g(t) = (\varphi_t^{-1})^*\tilde{g}(t)$ is a short time solution of (1) with the same initial data. A similar trick was invented earlier to break the diffeomorphism invariance in the case of Einstein's equations (e.g., see [2, 3] and [14]), through the use of harmonic coordinates or harmonic conditions relative to a background metric, but this trick is carried out in a 4-dimensional context. It would be interesting to find a trick which works entirely within the 3 + 1 context.

Our major goal is to show that under certain fairly general circumstances of cosmological interest in the early inflationary universe, solutions of the Ricci flow equation (1), with an initial metric of positive Ricci curvature, come quite close to solving not only the Einstein evolution equation (2), but also the constraints (3) and (4), provided the diffusion constant k, the dilation constant D and the cosmological constant Λ are related by $\Lambda = 3/16\,(c/k)^2$ (or $k = 1/4\,c\sqrt{3/\Lambda}$) and $D = 1/2\,c^2/k = 2c\sqrt{\Lambda/3}$. With regard to producing roundness, best results are obtained if $kD^{-1} = 2(k/c)^2 = 3/8\,\Lambda^{-1}$ is chosen to be close to, but strictly less than, the maximal value τ_m for which a solution of $h'(\tau) = -2\,Ric(h(\tau))$, $h(0) = g(0)$ exists. See Theorem 2 in Section 3 for a precise statement. Roughly, the closer $3/8\,\Lambda^{-1}$ is to τ_m, the more closely $(M, g(t))$ becomes a sphere (and the more closely $-c^2\,dt^2 + g(t)$ approximates the de Sitter solution) for large t, regardless of how inhomogeneous $g(0)$ is. It should be mentioned that the assumption that the initial metric $g(0)$ has positive Ricci curvature, is weaker than requiring that $g(0)$ have positive sectional curvature. Indeed, at a point, relative to an orthonormal frame for which Ric is diagonal, we have $R_{11} = R_{1212} + R_{1313}$ and cyclic permutations of this. Thus,

$R_{1212} = 1/2\,(R_{11} + R_{22} - R_{33})$, and negative sectional curvatures occur if $R_{33} > R_{11} + R_{22}$. There are some results (e.g. [10, 15]) in general relativity with a positive cosmological constant, concerning the asymptotic approach of *homogeneous* cosmological models, at least locally, to a de Sitter solution. However, even under the severe homogeneity assumption, the initial anisotropies will eventually reappear within the horizon, after (perhaps unobservably long after) the cosmological constant becomes small. In other words, unlike a suitable diffusive inflation, initial anisotropy is still present at large time on a *global* scale in these models.

In Section 2, we introduce some notation, recall some of Hamilton's results, and carry out some computations in preparation for the statements and proofs of Theorems 2 and 3 in Section 3. Some philosophical remarks and directions for further investigation are given in Section 4. In what follows, we work in the C^∞ (smooth) category. The author gratefully acknowledges Richard Hamilton for his monumental groundwork and for interesting conversations and seminars which made this article possible.

2. PRELIMINARY RESULTS

PROPOSITION 1 (Hamilton [7]). *Let g_0 be a metric on a compact 3-manifold M. Then there exists a unique family $h(\tau)$ of τ-dependent metrics on M, defined for some maximal interval, $0 \leqslant \tau < \tau_m \leqslant \infty$ and smooth on $M \times [0, \tau_m)$, such that $h(0) = g_0$ and*

$$h'(\tau) = -2\,Ric(h(\tau)). \tag{6}$$

Moreover, if $Ric(g_0) > 0$ (i.e. $Ric(g_0)$ is positive definite at all points of M), then $Ric(h(\tau)) > 0$ for all $0 \leqslant \tau < \tau_m$, and $\tau_m < \infty$, and $\max\{R(h(\tau))_x : x \in M\} \to \infty$ as $\tau \to \tau_m$. ■

PROPOSITION 2. *Let g_0, $h(\tau)$ and τ_m be as in Prop. 1. Let D be a constant with $kD^{-1} \leqslant \tau_m$. Then the solution of the problem*

$$g'(t) = -2k\,Ric(g(t)) + D\,g(t), \quad g(0) = g_0 \tag{7}$$

exists for all time $t \geqslant 0$ and is given by

$$g(t) = e^{Dt}\,h[k[1 - e^{-Dt}]D^{-1}] = e^{Dt}\,h(\tau(t)). \tag{8}$$

Proof. Since the Ricci tensor is dilation-invariant, we have $Ric(h(\tau(t))) = Ric(g(t))$ and

$$g'(t) = e^{Dt}h'[k[1 - e^{-Dt}]D^{-1}]\,ke^{-Dt} + De^{Dt}h[kD^{-1}[1 - e^{-Dt}]]$$

$$= -2k\, Ric(h(\tau(t))) + Dg(t) = -2k\, Ric(g(t)) + Dg(t). \qquad \square$$

For motivation, we first consider the case where $(M, g(0))$ is 3-sphere of constant curvature. Let $d\sigma^2$ be the metric for a 3-sphere S^3 of radius 1, and let $h(\tau) = r(\tau)^2 d\sigma^2$. Then the equation $h'(\tau) = -2\, Ric(h(\tau))$ becomes $[r(\tau)^2]'\, d\sigma^2 = -4\, d\sigma^2$. Thus, $h(\tau) = [{r_0}^2 - 4\tau]\, d\sigma^2$. Then $\tau_m = 1/4\, {r_0}^2$. By Prop. 2, the corresponding solution of (7) is

$$(9) \qquad g(t) = e^{Dt}\, h[k[1 - e^{-Dt}]D^{-1}] = e^{Dt}[{r_0}^2 - 4kD^{-1}[1 - e^{-Dt}]]d\sigma^2$$
$$= [e^{Dt}{r_0}^2 - 4kD^{-1}[e^{Dt} - 1]]\, d\sigma^2 = [4kD^{-1} + e^{Dt}({r_0}^2 - 4kD^{-1})]d\sigma^2.$$

For an initial perfect 3-sphere of radius ρ_0, the two solutions $-c^2\, dt^2 + g_e(t)$ of Einstein's vacuum equations (2), (3) and (4), with $g = g_e$ and cosmological constant Λ, are given by

$$g_e(t) = {\rho_0}^2 \left(\frac{1}{2}\, (3/\Lambda)\rho_0^{-2} + \left[1 - \frac{1}{2}\, (3/\Lambda)\rho_0^{-2}\right] \cosh(2\sqrt{\Lambda/3}\, ct) \right.$$
$$\left. \pm (1 - (3/\Lambda)\rho_0^{-2})^{1/2} \sinh(2\sqrt{\Lambda/3}\, ct) \right) d\sigma^2$$

For $g_e(t)$ to exist we need $1 - (3/\Lambda)\rho_0^{-2} \geqslant 0$ or ${\rho_0}^2 \geqslant 3/\Lambda$. The initial velocity is $g_e'(0) = \pm\, 2c\rho_0(1/3\, \Lambda\rho_0^2 - 1)^{1/2}\, d\sigma^2$. For large t, we obtain, modulo exponentially decaying terms,

$$g_e(t) \underset{t\to\infty}{\sim} G_e(t) \equiv \left(\frac{1}{2}\, (3/\Lambda) + {\rho_0}^2 \left\{ \frac{1}{2} \left[1 - \frac{1}{2}\, (3/\Lambda)\rho_0^{-2}\right] \right.\right.$$
$$\left.\left. \pm \frac{1}{2}\, [1 - (3/\Lambda)\rho_0^{-2}]^{1/2} \right\} \exp(2\sqrt{\Lambda/3}\, ct) \right) d\sigma^2.$$

From this, we see that in order for $G_e(t)$ to match $g(t)$ in (9), it is necessary that $4kD^{-1} = \frac{1}{2}\, (3/\Lambda)$ and $D = 2c\sqrt{\Lambda/3}$. Thus, $k = \frac{1}{8}\, (3/\Lambda)D = \frac{1}{4}\, (3/\Lambda)\, c\sqrt{\Lambda/3} = \frac{1}{4}\, c\sqrt{3/\Lambda}$ or $16k^2 = 3c^2/\Lambda$. Hence,

$$(10) \qquad \Lambda = \frac{3}{16}\, (c/k)^2 = \frac{3}{8}\, D/k, \;\; D = \frac{1}{2}\, c^2/k = 2c\sqrt{\Lambda/3},$$

$$k = \frac{1}{2}\, c^2/D = \frac{1}{4}\, c\sqrt{3/\Lambda}, \quad c^2 = 2kD.$$

With $4kD^{-1} = \frac{1}{2}\, (3/\Lambda)$, it is easy to verify that for each r_0 with $r_0^2 - 4kD^{-1} > 0$ (i.e., $r_0^2 > \frac{1}{2}\, (3/\Lambda)$), there is a unique $\rho_0 > 0$, such that

$$(11) \qquad {r_0}^2 - 4kD^{-1} = {\rho_0}^2 \{ \frac{1}{2}\, [1 - 1/2(3/\Lambda)\rho_0^{-2}] \pm \frac{1}{2}\, [1 - (3/\Lambda)\rho_0^{-2}]^{1/2} \}$$

(i.e., $g(t) \equiv G_e(t)$), namely, choose $\rho_0^2 = r_0^2 + \frac{1}{8}(3/\Lambda)/(\frac{2}{3}\Lambda r_0^2 - 1)$, and use the negative initial velocity $-2c\rho_0(1/3\ \Lambda\rho_0^2 - 1)^{1/2}\, d\sigma^2$ if $r_0 < 3/4\ (3/\Lambda)$, and the positive initial velocity $2c\rho_0\ (1/3\ \Lambda\rho_0^2 - 1)^{1/2}\, d\sigma^2$ if $r_0 > 3/4\ (3/\Lambda)$ [of course, $g_e'(0) = 0$ for $r_0 = 3/4\ (3/\Lambda)$]. Conversely, given ρ_0 with $\rho_0^2 \geqslant \Lambda/3$, and a sign $\pm$ for the initial velocity $g_e'(0)$ if $\rho_0^2 > \Lambda/3$, we obtain $g(t) \equiv G_e(t)$ if

$$(12) \qquad r_0^2 = [r_0(\pm)]^2 \equiv \frac{1}{2}\rho_0^2\left(\left[1 + \frac{1}{2}(3/\Lambda)\rho_0^{-2}\right] \pm [1 - (3/\Lambda)\rho_0^{-2}]^{1/2}\right)$$

In summary, we have shown the following.

THEOREM 1. *For any $k > 0$, let $g(t; k, r_0) = r(t; k, r_0)^2\, d\sigma^2$ be the solution of*

$$g'(t) = -2k\, Ric(g(t)) + \frac{1}{2}c^2/k\, g(t),\ g(0) = r_0^2\, d\sigma^2$$

and let $g_e(t; k, \rho_0, \pm) = \rho(t; k, \rho_0, \pm)^2\, d\sigma^2$ be the solution of the empty-space Einstein equations (2), (3) and (4), with cosmological constant, $\Lambda = 3/16\ (c/k)^2$, and initial square radius $\rho_0^2 \geqslant 3/\Lambda$ and initial velocity $g_e'(0) = \pm 2c\rho_0(1/3\ \Lambda\rho_0^2 - 1)^{1/2}\, d\sigma^2$. Let $r_0(\pm) > 0$ be given by (12). Then,

$$|\rho(t; k, \rho_0, \pm)^2 - r(t; k, r_0(\pm))^2| \leqslant C\exp(-2\sqrt{\Lambda/3}\, ct),$$

for some constant C independent of t. ■

REMARK. More precisely, it is straightforward to show that

$$(13) \qquad \rho(t; k, \rho_0, \pm)^2 - r(t; k, r_0(\pm))^2 = [r_0(\mp)]^2 \exp(-2\sqrt{\Lambda/3}\, ct).$$

Note that both terms on the left hand side of (13) *grow* exponentially with t, and yet their difference *decays* exponentially. Also, note that when $\Lambda r_0^2 \gg 1$ (or $\Lambda\rho_0^2 \gg 1$ with positive initial velocity), then $r_0(+) \approx \rho_0$ by (12), and $[r_0(-)]^2 \approx 1/2\ (3/\Lambda) \ll [r_0(+)]^2$. Thus, when $\Lambda r_0^2 \gg 1$, (13) indicates a close relative agreement for *all* $t \geqslant 0$. While agreement for large t is always good, the disparity for small t is unbounded as r_0^2 approaches the lower bound $1/2\ (3/\Lambda)$, since $\rho_0^2 = r_0^2 + 1/8\ (3/\Lambda)/(2/3\ \Lambda r_0^2 - 1)$. □

In Theorems 2 and 3 of Section 3, we do not assume perfect spherical symmetry, but rather that $g(0)$ is a metric with positive-definite Ricci tensor. We will need the ensuing results to handle this. The following equation is similar to (6), but it produces metrics $\tilde{h}(\tilde{\tau})$ which are normalized in the sense that the volume of $(M, \tilde{h}(\tilde{\tau}))$ is constant:

(14) $$\tilde{h}'(\tilde{\tau}) = -2\left(Ric(\tilde{h}(\tilde{\tau})) - \frac{1}{3}\,\bar{R}(\tilde{h}(\tilde{\tau}))\,\tilde{h}(\tilde{\tau})\right)$$

where

$$\bar{R}(h(\tau)) = \left[\int_M dv(\tilde{h}(\tilde{\tau}))\right]^{-1} \int_M R(\tilde{h}(\tilde{\tau}))\,dv(\tilde{h}(\tilde{\tau}))$$

is the average scalar curvature of $\tilde{h}(\tilde{\tau})$. Note that volume is preserved, since

$$2\,\frac{d}{d\tilde{\tau}} \int_M d\tilde{v}(\tilde{h}(\tilde{\tau})) = \int_M Tr[\tilde{h}'(\tilde{\tau})]\,\tilde{v}(\tilde{h}(\tilde{\tau}))$$

$$= -2 \int_M \left(Tr[Ric(\tilde{h}(\tilde{\tau}))] - \frac{1}{3}\,\bar{R}(\tilde{h}(\tilde{\tau}))\,Tr\,\tilde{h}(\tilde{\tau})\right)\tilde{v}(\tilde{h}(\tilde{\tau}))$$

$$= -2 \int_M \left(R(\tilde{h}(\tilde{\tau})) - \frac{1}{3}\,\bar{R}(\tilde{h}(\tilde{\tau}))3\right)\tilde{v}(\tilde{h}(\tilde{\tau})) = 0.$$

It is not hard to get solutions $h(\tau)$ of (14) from solutions $h(\tau)$ of (6). Indeed, we have

PROPOSITION 3. *Let $h(\tau)$ and τ_m be as in Prop. 1. With*

$$\psi(\tau) \equiv \exp\left(\frac{2}{3}\int_0^{\tau} \bar{R}(h(\sigma))\,d\sigma\right)$$

and

$$\tilde{\tau}(\tau) \equiv \int_0^{\tau} \psi(\sigma)\,d\sigma,$$

let

(15) $$\tilde{h}(\tilde{\tau}) \equiv \psi(\tau)\,h(\tau).$$

Then $h(\tau)$ solves (14) for $0 \leqslant \tilde{\tau} < \tilde{\tau}(\tau_m)$.

Proof.

$$\begin{aligned}\tilde{h}'(\tilde{\tau}) &= [\psi(\tau)\, h'(\tau) + \psi'(\tau)\, h(\tau)]\, \tau'(\tilde{\tau}) \\ &= [\psi(\tau) h'(\tau) + \psi'(\tau)\, h(\tau)]\, \psi(\tau)^{-1} \\ &= h'(\tau) + \psi'(\tau)\psi(\tau)^{-1}\, h(\tau) = -2\, Ric(h(\tau)) \\ &+ \frac{2}{3}\, R(h(\tau))\, h(\tau) = -2\, Ric\,(\tilde{h}(\tilde{\tau})) + \frac{2}{3}\, R(\tilde{h}(\tilde{\tau}))\, \tilde{h}(\tilde{\tau}).\end{aligned}$$ ∎

For an integer $n \geq 0$, there are many different, yet topologically equivalent, C^n norms on the linear space of tensors of a given type on M. For definiteness, let G be any fixed C^∞ Riemannian metric on M, and let ∇ be the Levi-Civita connection of G. For any tensor T and integer $n \geq 0$, we define the C^n norm $\| T \|_n$ by

$$\| T \|_n \equiv \max_{x \in M} \left[\sum_{i=0}^{n} \tilde{G}_x (\nabla^i T, \nabla^i T)^{1/2} \right], \tag{16}$$

where $\tilde{G}_x$ is the inner product induced by G_x on tensors at a point $x \in M$, and $\nabla^i T$ is the i-fold application of ∇ to T. For a different choice of metric, we obtain an equivalent C^n norm. Recall that norms $\| \ \|$ and $\| \ \|'$ on a vector space V are equivalent if, for some constant C, $C^{-1} \| v \| \leq \| v \|' \leq C \| v \|$ for all $v \in V$. For definiteness, one may choose $G = h(0)$, in all that follows. The following deep result is a direct consequence of results in Section 17 of [7].

PROPOSITION 4 (Hamilton). *Let $h(\tau)$ and $\tilde{h}(\tilde{\tau})$ be as in Prop. 3. If $Ric(h(0)) > 0$, then $\tilde{\tau}(\tau_m) = \infty$ (i.e., $\tilde{h}(\tilde{\tau})$ exists for all $\tilde{\tau} \geq 0$), and $\tilde{h}(\tilde{\tau})$ converges to a metric, say $\tilde{h}_\infty$, of constant curvature (e.g., M must be diffeomorphic to a sphere if M is simply-connected). The convergence is in C^n for each $n \geq 0$. Indeed, $\| \tilde{h}(\tilde{\tau}) - \tilde{h}_\infty \|_n \leq A_n e^{-\delta_n \tilde{\tau}}$, for some positive constants A_n and δ_n, where $\| \ \|_n$ is defined by (16). In particular, since $\tilde{h}_\infty$ has constant curvature, we have $Ric(\tilde{h}_\infty) - 1/3\ R(\tilde{h}_\infty)\tilde{h}_\infty = 0$, and thus for some positive constants A and δ, depending on n.*

$$\| Ric(\tilde{h}(\tilde{\tau})) - \frac{1}{3}\, R(\tilde{h}(\tilde{\tau}))\, \tilde{h}(\tilde{\tau}) \|_n \leq A e^{-\delta \tilde{\tau}}. \tag{17}$$

More generally, if $P(h)$ is a tensor which is formed, by means of tensor product and/or contraction, from the metric h and the curvature tensor and/or its covariant derivatives, then

$$\| P(\tilde{h}(\tilde{\tau})) - P(\tilde{h}_\infty) \|_n \leq A e^{-\delta \tilde{\tau}}, \tag{18}$$

for some constants A and δ, depending on P, n, and $\tilde{h}(0)$.

PROPOSITION 5. *If the metric g(t) satisfies* $g'(t) = -2k\, Ric(g(t)) + Dg(t)$, *then*

$$(19)\qquad \frac{d}{dt}[Ric(g)] = k[\Delta Ric(g) - 6\, Ric(g) \times Ric(g)$$

$$+ 3R(g)\, Ric(g) + (2\| Ric(g)\|^2 - R(g)^2)g].$$

Proof. By Prop. 2, $g(t) = e^{Dt} h[k[1 - e^{-Dt}]D^{-1}] = e^{Dt}h(\tau(t))$. Since the Ricci tensor is invariant under a spatially constant conformal change of the metric, we have $Ric(g(t)) = Ric(h(\tau))$. According to a full derivation in [7], we have (where Δ, x and $\| \;\|$ are computed with respect to $h(\tau)$)

$$\frac{d}{d\tau}[Ric(h(\tau))] = \Delta Ric(h) - 6\, Ric(h) \times Ric(h)$$

$$+ 3R(h)\, Ric(h) + (2\| Ric(h)\|^2 - R(h)^2)\, h$$

Thus, $[Ric(g(t))]' = [Ric(h(\tau))]'\, \tau'(t) = [Ric(h(\tau))]'\, ke^{-Dt}$. This means that (19) holds, where Δ, x and $\| \;\|$ are computed with respect to $g(t)$. ■

As we will see, the equations (20), (21) and (22) in the Propositions 6 and 7 below are primarily responsible for the large-time agreement of diffusive inflation and general relativity, when the initial metric has positive Ricci curvature. Notice that the right hand side of (20) has a factor of $(k/c)^2$ (of dimension length-squared) modulo the term $-1/2[R(g)g - 3\, Ric(g)]$ which vanishes on the sphere. Similarly, the right hand sides of (21) and (22) have factors of $(k/c)^2$ and (k/c), respectively. For dimensional reasons, factors of $(k/c)^2$ must multiply terms which are essentially *inversely* proportional to an exponentially growing metric of dimension length-squared. Hence these terms decay exponentially, as we prove precisely in Section 3.

PROPOSITION 6. *If the metric g(t) satisfies* $g'(t) = -2k\, Ric(g(t)) + 1/2\, c^2/k\; g(t)$, *then*

$$(20)\qquad c^{-2}g'' - c^{-2}\left[g' \times g' - \frac{1}{2}\, Tr(g')g'\right] + 2\, Ric(g) - \frac{3}{8}\,(c/k)^2 g$$

$$= -2(k/c)^2[\Delta Ric(g) - 4\, Ric(g) \times Ric(g)$$

$$+ 2R(g)\, Ric(g) + (2\| Ric(g)\|^2 - R(g)^2)g] - \frac{1}{2}[R(g)g - 3Ric(g)].$$

Proof. Let $g'(t) = -2k\ Ric(g(t)) + Dg(t)$. We set $D = 1/2\ c^2/k$ later. Then

$$g' \times g' - 1/2\ Tr(g')g'$$
$$= [-2k\ Ric(g) + Dg] \times [-2k\ Ric(g) + Dg]$$
$$-\frac{1}{2}\ Tr[-2k\ Ric(g) + Dg]\,[-2k\ Ric(g) + Dg]$$
$$= 4k^2\ Ric(g) \times Ric(g) - 4kD\, g \times Ric(g) + D^2 g \times g$$
$$+\frac{1}{2}\ [[-4k^2 R(g)\, Ric(g) + 2kD\, R(g)\, g] + 3[2kD\ Ric(g) - D^2 g]]$$
$$= 4k^2\ Ric(g) \times Ric(g) - 4kDg \times Ric(g) + D^2 g$$
$$-2k^2\ R(g)\, Ric(g) + kD\, R(g)g + 3kD\ Ric(g) - \frac{3}{2}\ D^2 g.$$

In the following, we have used Prop. 5 to compute $[Ric(g(t))]'$.

$$g'' = -2k\ [Ric(g)]' + Dg' = -2k\ [Ric(g)]' + D[-2k\,Ric(g) + Dg]$$
$$= -2k^2\,[\Delta\, Ric(g) - 6\ Ric(g) \times Ric(g) + 3R(g)\, Ric(g)$$
$$+ (2\,\|\, Ric(g)\,\|^2 \quad R(g)^2)g] - 2kD\ Ric(g) + D^2 g$$

Combining the above results, we have

$$c^{-2}g'' - c^{-2}\left[g' \times g' - \frac{1}{2}\ Tr(g')g'\right] + 2\ Ric(g) - 2\Lambda g \tag{20'}$$
$$= -2(k/c)^2\ [\Delta\, Ric(g) - 6\ Ric(g) \times Ric(g) + 3R(g)\ Ric(g)$$
$$+ (2\,\|\, Ric(g)\,\|^2 - R(g)^2)\, g] - 2D(k/c^2)\ Ric(g)$$
$$+ (D/c)^2\ g - c^{-2}\Big(4k^2\ Ric(g) \times Ric(g) - 4kDg \times Ric(g) + D^2 g$$
$$-2k^2\ R(g)\, Ric(g) + kD\, R(g)g + 3kD\ Ric(g) - \frac{3}{2}\ D^2 g\Big)$$
$$+ 2\ Ric(g) - 2\Lambda g$$
$$= -2(k/c)^2\ [\Delta\, Ric(g) - 4\ Ric(g) \times Ric(g) + 2\ R(g)\ Ric(g)$$
$$+ (2\,\|\, Ric(g)\,\|^2 - R(g)^2)\, g] - Dk/c^2\ R(g)g$$

(20') $\qquad + [2 - Dk/c^2]\, Ric(g) + [3/2\, (D/c)^2 - 2\Lambda] g.$

Upon setting $D = 1/2\, c^2/k$ and $\Lambda = 3/16\, (c/k)^2$, we obtain (20). ∎

REMARK. The term in expression (20') with the factor of $(k/c)^2$ will be seen (in Section 3) to dacay exponentially with time, for $kD^{-1} < \tau_m$. Recalling the Einstein evolution equation (2), the above proof then reveals the necessity of setting $2\Lambda = 3/2\, (D/c)^2$, if the evolution (2) is to be nearly satisfied, since otherwise $[3/2\, (D/c)^2 - 2\Lambda] g$ in (20') grows exponentially, as we will also see. While it turns out that $-Dk/c^2\, R(g) g + [2 - Dk/c^2]\, Ric(g)$ in (20') at least remains bounded, its trace is $[2 - 4Dk/c^2]\, R(g)$, which suggests that $D = 1/2\, c^2/k$ is the optimal choice for D, as we already found in the case of spherical symmetry. □

PROPOSITION 7. *If the metric $g(t)$ satisfies $g'(t) = -2k\, Ric(g(t)) + 1/2\, c^2/k\, g(t)$, then*

$$\frac{1}{4} c^{-2} [Tr(g')^2 - \| g' \|^2] + R(g) - \frac{3}{8} (c^2/k^2) \tag{21}$$

$$= (k/c)^2\, [R(g)^2 - \| Ric(g) \|^2]$$

and

$$c^{-1}\, Div[g' - Tr(g') g] = (k/c)\, d[R(g)]. \tag{22}$$

Proof. Let $g'(t) = -2k\, Ric(g(t)) + D\, g(t)$. We set $D = 1/2\, c^2/k$ later. For (21), we have

$$\frac{1}{4} c^{-2} [Tr(g')^2 - \| g' \|^2]$$

$$= \frac{1}{4} c^{-2} [Tr(-2k\, Ric(g) + Dg)^2 - \| -2k\, Ric(g) + Dg \|^2]$$

$$= (k/c)^2 \left(\left[-R(g) + \frac{3}{2} D/k \right]^2 - \left\| -Ric(g) + \frac{1}{2} D/k\, g \right\|^2 \right.$$

$$= (k/c)^2 \left(R(g)^2 - 3D/k\, R(g) + \left(\frac{3}{2} D/k \right)^2 \right.$$

$$\left. - \left[\| Ric(g) \|^2 - D/k\, R(g) + \frac{3}{4} (D/k)^2 \right] \right)$$

$$= (k/c)^2 \left(R(g)^2 - 2D/k\, R(g) - \| Ric(g) \|^2 + \frac{3}{2} (D/k)^2 \right)$$

$$= (k/c)^2 \, [R(g)^2 - \| Ric(g) \|^2] + \frac{3}{2} (D/c)^2 - 2Dk/c^2 \, R(g).$$

$$\left(\text{Now set } D = \frac{1}{2} c^2/k. \right)$$

For (22), we have, where we use the contracted Bianchi identity

$$Div \left[- Ric(g) + \frac{1}{2} R(g) g \right] = 0,$$

$$Div[g' - Tr(g')\, g]$$

$$= Div[(-2k\, Ric(g) + Dg) - Tr(-2k\, Ric(g) + Dg)\, g]$$

$$= 2k\, Div[- Ric(g) + R(g)g] = 2k\, Div\left[- Ric(g) + \frac{1}{2} R(g)g \right.$$

$$\left. + \frac{1}{2} R(g)g \right] = 2k\, Div\left[\frac{1}{2} R(g)\, g \right] = k\, d[R(g)]. \qquad ■$$

3. MAIN THEOREMS FOR THE INHOMOGENEOUS CASE

Recall that Theorem 1 shows that every solution of (23) below in which (M, g_0) is a perfect S^3 of radius r_0 with $\tau_m = 1/4\, r_0{}^2 > 1/8\, (3/\Lambda) = 2(k/c)^2$ exponentially approaches the de Sitter solution of Einstein's equations with cosmological constant $\Lambda = 3/16\, (c/k)^2$ as $t \to \infty$. Here, we consider the case where g_0 is only assumed to have positive Ricci curvature. It is not true in general that the solution $g(t)$ approaches a perfect de Sitter solution as $t \to \infty$. However, the proofs below show that by choosing k such that $2(k/c)^2$ is sufficiently close to (but less than) τ_m, we can guarantee that any preassigned degree of roundness (spatial constancy of sectional curvature) for $g(t)$ can be achieved as $t \to \infty$. Modulo this deviation from roundness, the Einstein evolution equation (2) is satisfied to within exponentially decaying terms. Moreover, the constraints (3) and (4) are satisfied modulo exponentially decaying terms, in spite of any residual deviation of $g(t)$ from roundness.

THEOREM 2. *Let (M, g_0) be a compact Riemannian 3-manifold with positive Ricci curvature tensor. For any $\epsilon > 0$, there is a heat diffusivity constant $k > 0$,* *such that the solution g(t) of*

$$g'(t) = -\ 2k\, Ric(g(t)) + \frac{1}{2}(c^2/k)\, g(t) \text{ with } g(0) = g_0 \tag{23}$$

exists for all $t > 0$, *and eventually satisfies, to within error* ϵ, *the empty-space Einstein evolution equation (2) with cosmological constant* $\Lambda = 3/16\,(c/k)^2$. *Moreover, the constraints are satified modulo terms which decay exponentially with time. More precisely, for any integer* $n \geqslant 0$, *there are constants* C_1 *and* C_2 *and a time T, such that for all* $t \geqslant T$,

$$\Big\| c^{-2}g''(t) - c^{-2}\Big[g'(t)\times g'(t) - \frac{1}{2}\, Tr(g'(t))\, g'(t)\Big] + 2\, Ric(g(t)) - \frac{3}{8}\,(c/k)^2\, g(t) \Big\|_n \leqslant \epsilon, \tag{24}$$

$$\Big\|\frac{1}{4}c^{-2}[Tr(g'(t))^2 - \| g'(t)\|^2] + R(g(t)) - \frac{3}{8}\,(c^2/k^2)\Big\|_n \leqslant C_1 e^{-c^2 t/k} \tag{25}$$

$$\| c^{-1}\, Div[g'(t) - Tr(g'(t))\, g(t)]\|_n \leqslant C_2 e^{-1/2c^2 t/k} \tag{26}$$

The constants ϵ *and* C_2 *can be made arbitrarily small by choosing k such that* $2(k/c)^2$ *is sufficiently close to, but less than,* τ_m *(see Prop. 1 for the definition of* τ_m*).*

Proof. As in Prop. 1, let $h(\tau)$ $(0 \leqslant \tau < \tau_m)$ solve $h'(\tau) = -2\, Ric(h(\tau))$. Then by Prop. 2,

$$g(t) = e^{1/2c^2 t/k}\, h[2(k/c)^2[1 - e^{-1/2c^2 t/k}]] \tag{27}$$

satisfies (7) for all $t \geqslant 0$, provided that $2(k/c)^2 \leqslant \tau_m$. Let $k < c\sqrt{\tau_m/2}$, so that this condition is strictly met. Let $\tau(t) \equiv 2(k/c)^2\ [1 - e^{-1/2c^2 t/k}]$, and note that $\tau(\infty) < \tau_m$ (strict). From Prop. 3, we have $\tilde{h}(\tilde{\tau}) = \psi(\tau)h(\tau)$, where

$$\psi(\tau) \equiv \exp\left(\frac{2}{3}\int_0^\tau \bar{R}(h(\sigma))\, d\sigma\right).$$

Thus,

$$g(t) = e^{1/2c^2 t/k}\, h(\tau(t)) = \psi(\tau(t))^{-1}\, e^{1/2c^2 t/k}\, \tilde{h}(\tilde{\tau}(\tau(t))), \tag{28}$$

where

$$\tilde{\tau}(\tau) \equiv \int_0^{\tau} \psi(\sigma)\, d\sigma.$$

For any metric G on M, let

(29) $$E[G] \equiv \Delta\, Ric(G) - 4\, Ric(G) \times Ric(G) + 2\, R(G)\, Ric(G) + (2\, \| Ric(G) \|^2 - R(G)^2)G.$$

From (28), we have

(30) $$E[g(t)] = \psi(\tau(t))\, e^{-1/2c^2 t/k}\, E[\tilde{h}(\tilde{\tau}(\tau(t)))].$$

Since $\tau(\infty) < \tau_m$, there is a constant K_1, such that $\psi(\tau(t)) \leqslant K_1 < \infty$ for all $t \geqslant 0$, and we also have $\tilde{\tau}(\tau(\infty))) < \infty$. Since $\tilde{\tau}(\tau(\infty))) < \infty$ and M is compact, there is a constant K_2, such that $\| E[\tilde{h}(\tilde{\tau}(\tau(t)))] \|_n \leqslant K_2$ for all $t \geqslant 0$. (Actually, we do not need the finiteness of $\tilde{\tau}(\tau(\infty))$ for this, because $\tilde{h}(\tilde{\tau})$ converges in all of the C^n norms as $\tilde{\tau} \to \infty$ by the difficult Prop. 4). Thus, for $K_3 = K_1 K_2$, using (30), we have

(31) $$\| E(g(t)) \|_n \leqslant K_3\, e^{-1/2c^2 t/k}$$

For any metric G on M, let $F[G] \equiv Ric(G) - 1/3\, R(G)G$. Note that $F(\lambda G) = F(G)$ for any constant $\lambda > 0$. Thus, $F[g(t)] = F[\tilde{h}(\tilde{\tau}(\tau(t)))]$. By (17) in Prop. 4, for any $\epsilon_1 > 0$, there is a constant K_4, such that $\| F(\tilde{h}(\tilde{\tau})) \|_n \leqslant \epsilon_1$ for all $\tilde{\tau} \geqslant K_4$. So far, the only restriction on k is that $2(k/c)^2 < \tau_m$. Now we also require that $\tilde{\tau}(2(k/c)^2) > K_4$, which will be true provided that $2(k/c)^2$ is sufficiently close to τ_m (i.e., we choose k such that $\tilde{\tau}^{-1}(K_4) < 2(k/c)^2 < \tau_m$). Let $T_1 \equiv \tau^{-1}(\tilde{\tau}^{-1}(K_4))$. Then $\| F[g(t)] \|_n = \| F(\tilde{h}(\tilde{\tau}(\tau(t)))) \|_n \leqslant \epsilon_1$, for all $t \geqslant T_1$. Thus, by (20) and (31), for all $t \geqslant T_1$,

$$\left\| c^{-2} g'' - c^{-2}\left[g' \times g' - \frac{1}{2}\, Tr(g')\, g' \right] + 2\, Ric(g) - \frac{3}{8}\, (c/k)^2 g \right\|_n$$

$$\leqslant \left\| -2(k/c)^2\, E(g) - \frac{3}{2}\, F(g) \right\|_n \leqslant 2(k/c)^2\, \| E(g) \|_n + \frac{3}{2}\, \| F(g) \|_n$$

$$\leqslant K_3\, 2(k/c)^2\, e^{-1/2c^2 t/k} + \frac{3}{2}\, \epsilon_1$$

Let T_2 be such that for $t \geqslant T_2$, we have $K_3\, 2(k/c)^2\, e^{-1/2c^2 t/k} \leqslant 1/2\, \epsilon$, and choose $\epsilon_1 < 1/3\, \epsilon$. Finally, let $T = \max\, (T_1, T_2)$. Then we have (24) for all $t \geqslant T$. The inequalities (25) and (26) are easily deduced in a similar fashion from (21) and (22), (18) in Prop. 4, and the following two equalities

$$R(g(t))^2 - \| Ric(g(t)) \|^2$$
$$= \psi(\tau(t))^2 e^{-c^2 t/k} [R(\tilde{h}(\tilde{\tau}))^2 - \| Ric(\tilde{h}(\tilde{\tau})) \|^2]|_{\tilde{\tau}=\tilde{\tau}(\tau(t))}$$
$$d[R(g)] = \psi(\tau(t))\, e^{-1/2c^2 t/k}\, d[R(\tilde{h}(\tilde{\tau}))]|_{\tilde{\tau}=\tilde{\tau}(\tau(t))}$$

Indeed, since $\| dR(\tilde{h}(\tilde{\tau})) \|_n \to 0$ as $\tilde{\tau} \to \infty$, the constant C_2 in (26) can be taken as small as desired by selecting k such that $2(k/c)^2$ is sufficiently close to (but less than) τ_m, and T sufficiently large. ∎

As it stands, Theorem 2 tells us that under suitable assumptions, $-c^2 dt^2 + g(t)$ nearly satisfies the Einstein field equations for large time. This does not immediately imply that $-c^2 dt^2 + g(t)$ is close to some exact solution of Einstein's equations (i.e., a nearly exact solution might not be near an exact solution). Nevertheless, in Theorem 2, by choosing ϵ sufficiently small and t sufficiantly large, $-c^2 dt^2 + g(t)$ eventually satisfies Einstain's equations to within any preassigned positive observational error, meaning that there is no obvious physical means of showing that it is not an exact solution. It may be possible to show that in general if one has an evolving metric $g(t)$ that satisfies the evolution (2) and the constraints (3) and (4) to within a sufficiently small error, then $-c^2 dt^2 + g(t)$ is close to an actual solution. We are unsure if this conjecture has been shown already, since most stability results involve perturbations from an exact solution and assume that the constraints are met exactly, but the failure of this conjecture might tarnish the theory of general relativity. In the special case at hand, we *can* prove that, for suitable choices, $-c^2\, dt^2 + g(t)$ is relatively near to an exact solution of (2), (3) and (4) for large t, namely a de Sitter solution. More precisely, we have

THEOREM 3. *With the assumptions of Theorem 2, assume also that M is simply-connected. Then M is diffeomorphic to a 3-sphere. Choose an integer $n \geq 0$, and let $\epsilon > 0$. For some metric $d\sigma^2$ of constant curvature 1 on M, some diffusion constant k, and some time $T \geq 0$, we have in the notation of Theorem 1,*

$$\| g(t) - \rho(t; k, \rho_0, \pm)^2\, d\sigma^2 \|_n \leq \epsilon \rho(t; k, \rho_0, \pm)^2, \text{ for all } t \geq T. \tag{32}$$

Proof. By Prop. 4, $\tilde{h}(\tilde{\tau})$ converges exponentially in C^n to a constant curvature metric, say $\tilde{h}(\infty)$, while maintaining constant volume. Since M is simply-connected, and M admits a metric of constant curvature, M must diffeomorphic to a 3-sphere. Thus, in the notation of Theorem 1, we may write $\tilde{h}(\infty) = r_0{}^2 d\sigma^2$. By Prop. 4, for any $\delta > 0$, we may choose some $\tilde{\tau}_0$, so that

$$\| \tilde{h}(\tilde{\tau}) - r_0{}^2\, d\sigma^2 \|_n \leq \delta, \text{ for all } \tilde{\tau} \geq \tilde{\tau}_0.$$

Choose k such that $\tilde{\tau}^{-1}(\tilde{\tau}_0) < 2(k/c)^2 < \tau_m$. Then, by (28), for $\tilde{\tau}(\tau(t)) \geq \tilde{\tau}_0$,

$$(33)\qquad \begin{aligned} &\| g(t) - e^{Dt}\psi(\tau)^{-1} r_0{}^2 d\sigma^2 \|_n \\ &= \| e^{Dt}\psi(\tau)^{-1}\tilde{h}(\tilde{\tau}) - e^{Dt}\psi(\tau)^{-1} r_0{}^2 d\sigma^2 \|_n \\ &= e^{Dt}\psi(\tau)^{-1} \| \tilde{h}(\tilde{\tau}) - r_0{}^2 d\sigma^2 \|_n \leqslant e^{Dt}\delta, \end{aligned}$$

since $\psi(\tau)^{-1} \leqslant 1$ (see Prop. 3). Let $\psi_0(\tau)^{-1} \equiv 1 - 4r_0{}^{-2}\tau$. In the notation of Theorem 1, note that $e^{Dt}\psi_0(\tau)^{-1} r_0{}^2 d\sigma^2 = r(t;\, k,\, r_0)^2 d\sigma^2$. Our strategy is to show that by choosing k, so that $2(k/c)^2$ is sufficiently close to τ_m, $\psi(\tau)^{-1}$ can be made close to $\psi_0(\tau)^{-1}$ for t sufficiently large. Then we show that (33) implies that $g(t)$ is relatively close to $r(t;\,k,\,r_0)^2 d\sigma^2$, and hence to $\rho(t;\,k,\,\rho_0,\,\pm)^2 d\sigma^2$ by Theorem 1. Recall from Prop. 3 that $h(\tau) = \psi(\tau)^{-1}\tilde{h}(\tilde{\tau})$, where

$$\psi(\tau)^{-1} = \exp\left[-\int_0^{\tau} \frac{2}{3}\, \overline{R}(h(\sigma))\, d\sigma\right].$$

Since we know more about $\tilde{h}(\tilde{\tau})$ than $h(\tau)$, we want to express $\psi(\tau)^{-1}$ as a function of $\tilde{\tau}$. By insisting that $h(\tau) = f(\tilde{\tau})\tilde{h}(\tilde{\tau})$ solve $h'(\tau) = -2\, Ric(h(\tau))$, one finds that $\psi(\tau)^{-1} = f(\tilde{\tau})$, where

$$(34)\qquad f(\tilde{\tau}) = \exp\left[-\int_0^{\tilde{\tau}} \frac{2}{3}\, \overline{R}(\tilde{h}(\tilde{\sigma}))\, d\tilde{\sigma}\right] \text{ and } \tau(\tilde{\tau}) = \int_0^{\tilde{\tau}} f(\tilde{\sigma})\, d\tilde{\sigma}.$$

By Prop. 4, $\tilde{h}(\tilde{\tau}) \to r_0{}^2 d\sigma^2$ in C^2, as $\tilde{\tau} \to \infty$. Thus, for any $\alpha > 0$, we can find $\tilde{\tau}_1$, such that $|\overline{R}(\tilde{g}(\tilde{\tau})) - 6r_0{}^{-2}| \leqslant 6r_0{}^{-2}\alpha$ for $\tilde{\tau} \geqslant \tilde{\tau}_1$. Consequently,

$$-4r_0{}^{-2}(1+\alpha)\,\tilde{\tau} \leqslant -\int_0^{\tilde{\tau}} \frac{2}{3}\, \overline{R}(\tilde{h}(\tilde{\sigma}))\, d\tilde{\sigma} \leqslant -4r_0{}^{-2}(1-\alpha)\,\tilde{\tau}$$

or

$$(35)\qquad \exp[-4r_0{}^{-2}(1+\alpha)\,\tilde{\tau}] \leqslant f(\tilde{\tau}) \leqslant \exp[-4r_0{}^{-2}(1-\alpha)\,\tilde{\tau}].$$

Since

$$\tau(\tilde{\tau}) = \int_0^{\tilde{\tau}} f(\tilde{\sigma})\, d\tilde{\sigma},$$

we obtain upon integrating (35) from 0 to $\tilde{\tau}$,

$$\frac{1}{4}\, r_0{}^2(1+\alpha)^{-1}\,[1 - \exp[-4r_0{}^{-2}(1+\alpha)\tilde{\tau}]]$$

(36) $$\leqslant \tau(\tilde\tau) \leqslant \frac{1}{4} r_0{}^2(1-\alpha)^{-1}[1-\exp[-4r_0{}^{-2}(1-\alpha)\tilde\tau]].$$

From (35) and (36), we then obtain

$$\frac{1}{4} r_0{}^2(1+\alpha)^{-1}[1-f(\tilde\tau)] \leqslant \tau(\tilde\tau) \leqslant \frac{1}{4} r_0{}^2(1-\alpha)^{-1}[1-f(\tilde\tau)]$$

or

$$|f(\tilde\tau) - [1-4r_0{}^{-2}\tau(\tilde\tau)]| \leqslant 4\alpha r_0{}^{-2}\tau(\tilde\tau)$$

or

(37) $$|\psi(\tau)^{-1} - \psi_0(\tau)^{-1}| \leqslant 4\alpha r_0{}^{-2}\tau < 4\alpha r_0{}^{-2}\tau_m,$$

for

$$\tilde\tau^{-1}(\tilde\tau_1) \leqslant \tau < \tau_m.$$

Choosing k so that $\tau_m > 2(k/c)^2 > \max\{\tilde\tau^{-1}(\tilde\tau_0), \tau^{-1}(\tilde\tau_1)\} \equiv \tau_3$ and using (33) and (37),

(38) $$\| g(t) - r(t;k,r_0)^2 d\sigma^2 \|_n = \| g(t) - e^{Dt}\psi_0(\tau)^{-1} r_0{}^2 d\sigma^2 \|_n$$
$$\leqslant \| g(t) - e^{Dt}\psi(\tau)^{-1} r_0{}^2 d\sigma^2 \|_n$$
$$+ \| e^{Dt}\psi(\tau)^{-1} r_0{}^2 d\sigma^2 - e^{Dt}\psi_0(\tau)^{-1} r_0{}^2 d\sigma^2 \|_n$$
$$\leqslant e^{Dt}\delta + e^{Dt} r_0{}^2 |\psi(\tau)^{-1} - \psi_0(\tau)^{-1}| \, \| d\sigma^2 \|_n$$
$$\leqslant e^{Dt}[\delta + 4\alpha\tau \| d\sigma^2 \|_n],$$

for $t > \tau^{-1}(\tau_3)$. Theorem 1 (or (13)) and (38) then yield

(39) $$\| g(t) - \rho(t;k,\rho_0,\pm)^2 \, d\sigma^2 \|_n \leqslant \| g(t) - r(t;k,r_0)^2 d\sigma^2 \|_n$$
$$+ \| r(t;k,r_0)^2 \, d\sigma^2 - \rho(t;k,\rho_0,\pm)^2 \, d\sigma^2 \|_n$$
$$\leqslant e^{Dt}[\delta + 4\alpha\tau_m \| d\sigma^2 \|_n + Ce^{-2Dt}], \text{ for } t > \tau^{-1}(\tau_3).$$

Now $\rho(t;k,\rho_0,\pm)^2 \geqslant \text{const.}\ e^{Dt}$, α and δ can be made arbitrarily small by choosing τ_0 and τ_1 sufficienlty large, and Ce^{-2Dt} is arbitrarily small for t sufficiently large. Thus, the desired result (32) follows from (39). ■

4. CONCLUDING REMARKS AND SPECULATIONS

Under certain circumstances (e.g., $\Lambda - 3/8\ \tau_m^{-1}$ small and positive) Theorems 2 and 3 exhibit the extent to which diffusive infaltion, with a wide variety of inhomogeneous initial conditions, eventually nearly satisfies the traditional Einstein theory with positive Λ, and nearly becomes a homogeneous de Sitter-like solution. Although our analysis suggests that the end results of relativistic inflation and of diffusive inflation are very much in agreement, a distinction between the two theories is that diffusive inflation is expected to produce somewhat more smoothing of inhomogeneities, since it is governed by a heat equation, as opposed to a wave equation. Some have concluded (e.g., see [1, 6, 11, 12]) that the usual inflationary theories, based on GUT symmetry-breaking, produce inhomogeneities which are from 1000 to 100,000 times larger than those observed in the background radiation. Crudely, the problem seems to be in dampening the density perturbations which arise in scenarios where a super-cooled universe, which is classically trapped in a false vacuum state, finally breaks its GUT (or other symmetry), via quantum mechanical tunneling to the true vacuum state. Some maintain that there are ways to fix this, and some say that it may not possible without introducing other problems (e.g., insufficient inflation, not enough reheating for baryogenesis, etc.). We refer the interested reader to the review articles [1, 6, 11, 12] and [13] and references therein for a survey of the efforts to construct viable theories of inflation. The consensus seems to be that a natural, uncontrived solution has yet to be found. Although it is not clearly motivated by current physical theory, diffusive inflation suggests a mathematically natural device to dampen density perturbations, and it might rescue some of the more natural inflationary models that have been scrapped. In this sense, diffusive inflation could serve as a useful device within existing (and possibly competing) version of inflation.

Of course, much needs to done in order to connect the pure geometry of diffusive geometry with the poorly understood (or perhaps, misunderstood) physical mechanisms which operate in the extreme circumstances characteristic of the inflationary era (e.g., at energy densities which exceed 10^{60} times that of the atomic nucleus). One might legitimately wonder if the situation is akin to trying to describe the quantum mechanics of the atom in terms of levers, pulleys, and gears, or even earth, water, fire and air. Perhaps one should explore the possibility that during the hypothetical inflationary era or before, the universe evolved according to (1), instead of (2), (3) and (4). In other words, perhaps during this era, geometry (and matter and gauge fields) evolved primarily by diffusion, instead of radiation. Given the remarkable connections which have arisen between relativity, quantum theory and thermodynamics, this does not seem so outlandish. Note that there is a very small Planck diffusion constant, namely $[G\hbar/c]^{1/2} = l_p^{\,2}/t_p$, the square of Planck length, divided by Planck time. The time that it would take for 50% of a point heat source to diffuse 1 cm is about 16,000 times the present age of the universe. Thus, currently it is very easy to overlook such a diffusion process. However, for a universe with diameter l_p, a point source could significantly diffuse throughout the entire universe in t_p. Nevertheless, the tenacy with which may hold on to relativistic invariance or locality is likely to prevent them from taking diffusive inflation very seriously. This is especially true for those who, in spite of actual experimental violation of Bell's inequality, would rather deny all forms of objective reality or contrafactual definiteness than give up locality. It is possible that *external* relativistic invariance might have emerged with the breaking of *internal* GUT symmetry? The fact that the constraints (3) and (4), which are consequences of relativistic invariance, are increasingly satisfied with time suggests tht perhaps relativistic invariance was not built in at the outset, but rather evolved. From a mathematical perspective, not only is (1) a simpler evolution equation to deal with than (2), but also one needs only to specify $g(0)$, instead of $g(0)$ *and* $g'(0)$, and the constraints (3) and (4) do not have to be deal with. Note that in accordance with the uncertainty principle, position and velocity cannot be separately specified, even modulo constraints, but rather their probability distributions are encoded in a single wave function. In diffusive inflation $g'(0)$ is encoded within $g(0)$, namely $g'(0) = -2k\ Ric(g(0)) + Dg(0)$. This may be a clue that perhaps diffusive inflation is a semi-classical effective manifestation of quantum gravity, providing a thin transitional boundary layer between quantum gravity and classical general relativity.

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